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AUTHOR(S):

Okada, Mari

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Free Boundary Problem for the Equation of One-Dimensional Motion of Compressible Viscous Gas

山口大・工　岡田真理(Mari Okada)

This is a joint work with Šárka MATUŠŮ - NEČASOVÁ and Tetu MAKINO.

1 Introduction

We investigate the equations

$$\begin{cases} \dfrac{\partial \rho}{\partial t}+\dfrac{\partial(\rho u)}{\partial \xi}=0, \\ \dfrac{\partial}{\partial t}(\rho u)+\dfrac{\partial}{\partial \xi}(\rho u^2+p)=\dfrac{\partial}{\partial \xi}\left(\mu\dfrac{\partial u}{\partial \xi}\right)-\rho g, \end{cases}$$

where $t>0$, $0<\xi<y(t)$. The unknown functions ρ and u represent the density and velocity, respectively; $p=a\rho^\gamma$ and $\mu=b\rho^\beta$ are the pressure and the viscosity coefficient , respectively, where a, b are positive constants and $\gamma>1$ and $0<\beta<\gamma-1$. The non-negative constant g is the gravitation constant; $\xi=0$ is the fixed boundary

$$u(t,0)=0,$$

and $\xi=y(t)$ is the free boundary, i.e. the interface of the gas and the vacuum:

$$\frac{dy}{dt}=u(t,y(t)) \quad \text{and} \quad \left(p-\mu\frac{\partial u}{\partial \xi}\right)(t,y(t))=0.$$

We want to show the global existence of a weak solutions and uniqueness. To prove it, we shall adopt the method of [1], [6] and use also some of the tools of paper [5].

Here, the main point in order to show the existence and the uniqueness of the solutions is the estimates to the solutions of the difference equations. Therefore we will show it in the next section.

We rewrite the equations in the Lagrangean mass coordinate:

$$x=\int_0^\xi \rho(t,\zeta)d\zeta.$$

Assuming that

$$\int_0^{y(t)} \rho(t,\xi)d\xi=1,$$

the above problem is transformed to the following fixed boundary problem;

$$\frac{\partial \rho}{\partial t}+\rho^2\frac{\partial u}{\partial x}=0, \tag{1.1}$$

$$\frac{\partial u}{\partial t} + \frac{\partial p}{\partial x} = \frac{\partial}{\partial x}\left(\mu\rho\frac{\partial u}{\partial x}\right) - g, \tag{1.2}$$

in $t > 0$ and $0 < x < 1$, where $p = a\rho^\gamma, \mu = b\rho^\beta$ with the boundary conditions

$$u(t,0) = 0, \quad \left(p - \mu\rho\frac{\partial u}{\partial x}\right)(t,1) = 0. \tag{1.3}$$

and the initial condition

$$(\rho, u)(0,x) = (\rho_0, u_0)(x), \quad 0 \le x \le 1. \tag{1.4}$$

In this paper we consider the following assumptions (A.1), (A.2) and (A.3) for the initial data and β

$$\rho_0 \in Lip[0,1] \text{ and } \rho_0(x) \ge \underline{\rho} \ (\underline{\rho} \text{ is a positive constant}), \tag{$A.1$}$$

$$u_0 \in C^1[0,1] \text{ and } \frac{du_0}{dx} \in Lip[0,1], \tag{$A.2$}$$

$$0 < \beta < \frac{1}{3}. \tag{$A.3$}$$

Definition :
A couple (ρ, u) is called a global weak solution for (1.1)-(1.4) if

$$\rho, u \in L^\infty([0,T] \times [0,1]) \cap C^1([0,T]; L^2(0,1)), \tag{1.5}$$

$$\rho^{\beta+1} u_x \in L^\infty([0,T] \times [0,1]) \cap C^{\frac{1}{2}}([0,T]; L^2(0,1)), \tag{1.6}$$

for any T, and the following equations hold:

$$\frac{\partial \rho}{\partial t} + \rho^2 \frac{\partial u}{\partial x} = 0, \tag{1.7}$$

for *a.e.* $x \in (0,1)$ and for any $t \ge 0$, and

$$\int_0^1 [\ \phi u_t - \phi_x(p - \mu\rho u_x) + \phi g\ ]\, dx = 0, \tag{1.8}$$

for any test function $\phi \in C_0^\infty((0,1])$ and for *a.e.* $t \in [0,T]$.

Remark 1 *Physicists claim that the viscosity of gas is proportional to the square root of the temperature (e.g. [2], vol.1, p.336). In this case, the temperature is keeping with $\rho^{\gamma-1}$, provided that the pressure p is proportional to the product of ρ and the temperature, i.e., the perfect fluid. In this situation we have $\beta = \dfrac{\gamma - 1}{2}$ and $\beta < \frac{1}{3}$ says $\gamma < \frac{5}{3}$.*

Remark 2 *The compressible and heat- conductive Navier-Stokes equations are obtained as the second approximation of the formal Chapman-Enskog expansion to the nonlinear Boltzmann equations for a rarefied simple gas. Here we assume the cut-off hard potentials (cf.[3]) and consider two important spacial cases: the hard sphere and the cut-off inverse power forces. Then the coefficient of viscosity is given explicitly, i.e. for the first case we have already mentioned in Remark 1, and for the second case, the viscosity is proportional to the power* $\frac{s+3}{2(s-1)}$ $(s \geq 5)$ *of the temperature (e.g.[4] p.103). Therefore in the case of the cut-off inverse power forces, we have* $\beta < \gamma - 1$ *says* $s > 5$, *provided that the equation of state is that of ideal and polytropic gas. From the condition* $\beta < \gamma - 1$ *(i.e.,* $s > 5$*) [4] deduced a plausible result by a mathematical rigorous way.*

2 Difference Scheme and Estimates

Discretizing the derivaties with respect to x of the equations (1.1) and (1.2), we have the following scheme:

$$\frac{d}{dt}\rho_{n-1} + {\rho_{n-1}}^2 \frac{u_n - u_{n-1}}{\Delta} = 0, \tag{2.1}$$

$$\frac{d}{dt}u_n + \frac{p_n - p_{n-1}}{\Delta} = \frac{1}{\Delta}\left[\mu_n\rho_n\frac{u_{n+1}-u_n}{\Delta} - \mu_{n-1}\rho_{n-1}\frac{u_n - u_{n-1}}{\Delta}\right] - g, \tag{2.2}$$

for $n = 1, 2, \ldots, N$, where $\Delta = \frac{1}{N}$, N being a large natural number which divides the interval [0,1] into N intervals with length Δ. We set

$$p_{n-1} = a{\rho_{n-1}}^{\gamma}, \tag{2.3}$$

$$\mu_{n-1} = b{\rho_{n-1}}^{\beta}. \tag{2.4}$$

The boundary conditions are

$$u_0(t) = 0, \quad \left(p_N - \mu_N\rho_N\frac{u_{N+1} - u_N}{\Delta}\right)(t) = 0. \tag{2.5}$$

and the initial conditions are

$$\rho_{n-1}(0) = \rho_0((n-1)\Delta) \geq \underline{\rho} > 0, \quad u_n(0) = u_0(n\Delta). \tag{2.6}$$

By the elementary theory of the ordinary differential equations, the Cauchy problem (2.1)-(2.6) admits a temporarily local solution in the domain $R^{2N} = \{(\rho_{n-1}, u_n)_{n=1,\ldots,N}\}$. Let $[0, T_\infty)$ be the right maximal interval of existence of this solution. By the equation (2.1) and the initial condition (2.6), we see $\rho_{n-1}(t) > 0$ for $0 < t < T_\infty$. We will prove that $T_\infty = +\infty$ after getting some a priori estimates.

First, we will show that the solution satisfies a priori estimates independent of Δ.

We set

$$y_n(t) = \sum_{k=1}^{n} \frac{\Delta}{\rho_{k-1}(t)}. \tag{2.7}$$

We get

Proposition 1 *Let (A1)-(A3) be satisfied then*

$$\frac{d}{dt} y_n(t) = u_n(t)$$

holds.

Proof. From the equation (2.1) and the boundary condition (2.5), we get

$$\dot{y}_n = -\sum_{k=1}^{n} \frac{\dot{\rho}_{k-1}}{\rho_{k-1}^2} \Delta = \sum_{k=1}^{n} (u_k - u_{k-1}) = u_n.$$

Next, we show the energy inequality.

Proposition 2 *There exists a constant C independent of t and Δ such that*

$$\sum_{n=1}^{N} \left(\frac{1}{2} {u_n}^2 + \frac{a}{\gamma - 1} {\rho_{n-1}}^{\gamma-1} + g y_n \right) (t) \Delta + \int_0^t \sum_{n=1}^{N} \left[\mu_{n-1} \rho_{n-1} \left(\frac{u_n - u_{n-1}}{\Delta} \right)^2 \right] (\tau) \Delta d\tau$$
$$= \sum_{n=1}^{N} \left(\frac{1}{2} {u_n}^2 + \frac{a}{\gamma - 1} {\rho_{n-1}}^{\gamma-1} + g y_n \right) (0) \Delta \ \leq \ C.$$

Proof. Multiplying the equation (2.2) by $u_n \Delta$, summing from $n = 1$ to $n = N$, using the boundary condition (2.5) and Proposition 1 and integrating with respect to τ from 0 to t, we have the required expression. Applying (2.6) and since ${\rho_0}^{\gamma-1}, u_0 \in C[0,1]$ we obtain the bound of the right hand side. (2.7) is obtained by the theory of Riemann integral.

From the above a priori estimates we have the following.
Lemma $T_\infty = +\infty$, that is, the solution of (2.1), (2.2) and (2.6) exists for $0 \leq t < +\infty$ and $\rho_{n-1} > 0$ for $0 \leq t < +\infty$.

Hereafter we consider estimates in an interval $0 \leq t \leq T$, where T is an arbitrarily fixed large number, and $C(T)$ denotes various constants depending on the parameters γ, β, g, a, b and the initial conditions ρ_0 and u_0, which does not depend on Δ.

Proposition 3 *The following inequality*

$$\rho_{n-1} \leq C(T)$$

is satisfied.

Proof.

Multiplying the equation (2.2) by Δ, summing over $k = n, \dots, N$ and using the boundary condition (2.5), the equation (2.1) and the relation (2.4), we get

$$\text{(2.8)} \qquad \sum_{k=n}^{N} \dot{u}_k \Delta - p_{n-1} + g(1 - n\Delta) = -\mu_{n-1}\rho_{n-1}\frac{u_n - u_{n-1}}{\Delta} = \frac{d}{dt}\left(\frac{b}{\beta}\rho_{n-1}{}^{\beta}\right).$$

Integrating (2.8) with respect to τ from 0 to t, we have

$$\frac{b}{\beta}\rho_{n-1}{}^{\beta}(t) = \frac{b}{\beta}\rho_{n-1}{}^{\beta}(0) - \int_0^t p_{n-1}(\tau)d\tau + \sum_{k=n}^{N}(u_k(t) - u_k(0))\Delta + g(1 - n\Delta)t.$$

Applying the assumption (A.1), using the positiveness of the density and Proposition 2, we obtain the required estimate.

Also we have the following Proposition

Proposition 4 *Under the assumptions (A.1)-(A.3) the inequality*

$$\sum_{n=1}^{N} \rho_{n-1}{}^{\beta-1}(t)\Delta \;\le\; C(T)$$

holds.

Proof.

Dividing the relation (2.8) by ρ_{n-1}, integrating with respect to τ from 0 to t, multiplying by Δ, summing from $n = 1$ to $n = N$ and using (2.3), we get

$$\begin{aligned} \frac{b}{1-\beta}\sum_{n=1}^{N}\rho_{n-1}{}^{\beta-1}(t)\Delta \;&=\; \frac{b}{1-\beta}\sum_{n=1}^{N}\rho_{n-1}{}^{\beta-1}(0)\Delta - \int_0^t \sum_{n=1}^{N}\frac{\Delta}{\rho_{n-1}(\tau)}\sum_{k=n}^{N}\dot{u}_k(\tau)\Delta d\tau \\ &\quad + a\int_0^t \sum_{n=1}^{N}\rho_{n-1}{}^{\gamma-1}(\tau)\Delta d\tau - g\int_0^t \sum_{n=1}^{N}\frac{1-n\Delta}{\rho_{n-1}(\tau)}\Delta d\tau. \end{aligned}$$

From (2.6) and Proposition 3 follow that the first term and the third one of the right hand side are bounded . Applying (2.7) we obtain that the forth one is negative .

Therefore if the second term of the right hand side can be estimated, we will obtain the required estimate. By (2.1), (2.5), (2.6) and Proposition 2, we have

$$\begin{aligned} &\int_0^t \sum_{n=1}^{N}\frac{\Delta}{\rho_{n-1}(\tau)}\sum_{k=n}^{N}\dot{u}_k(\tau)\Delta d\tau \\ &= \int_0^t \left[\frac{d}{d\tau}\left(\sum_{n=1}^{N}\frac{\Delta}{\rho_{n-1}(\tau)}\sum_{k=n}^{N}u_k(\tau)\Delta\right) - \sum_{n=1}^{N}\frac{d}{d\tau}\left(\frac{\Delta}{\rho_{n-1}(\tau)}\right)\sum_{k=n}^{N}u_k(\tau)\Delta\right]d\tau \\ &= \sum_{n=1}^{N}\frac{\Delta}{\rho_{n-1}(t)}\sum_{k=n}^{N}u_k(t)\Delta - \sum_{n=1}^{N}\frac{\Delta}{\rho_{n-1}(0)}\sum_{k=n}^{N}u_k(0)\Delta \\ &\qquad - \int_0^t \sum_{n=1}^{N}(u_n - u_{n-1})(\tau)\sum_{k=n}^{N}u_k(\tau)\Delta d\tau \end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=1}^{N}\left(\frac{\Delta}{\rho_{n-1}(t)}-\frac{\Delta}{\rho_{n-1}(0)}\right)\sum_{k=n}^{N}u_k(t)\Delta+\sum_{n=1}^{N}\frac{\Delta}{\rho_{n-1}(0)}\sum_{k=n}^{N}(u_k(t)-u_k(0))\Delta \\
&\qquad -\int_0^t\sum_{n=1}^{N}{u_n}^2(\tau)\Delta d\tau \\
&\le \sum_{n=1}^{N}\int_0^t\frac{d}{d\tau}\left(\frac{\Delta}{\rho_{n-1}(\tau)}\right)d\tau\sum_{k=n}^{N}u_k(t)\Delta+\sum_{n=1}^{N}\frac{\Delta}{\rho_{n-1}(0)}\frac{1}{2}\left[\left(\sum_{n=1}^{N}({u_n}^2(t)+{u_n}^2(0))\Delta\right)+1\right] \\
&\le \int_0^t\sum_{n=1}^{N}(u_n-u_{n-1})(\tau)\sum_{n=k}^{N}u_k(t)\Delta d\tau+C_1 \\
&= \int_0^t\sum_{n=1}^{N}u_n(\tau)u_n(t)\Delta d\tau+C_1 \\
&\le \int_0^t\left(\sum_{n=1}^{N}{u_n}^2(\tau)\Delta\right)^{\frac{1}{2}}d\tau\left(\sum_{n=1}^{N}{u_n}^2(t)\Delta\right)^{\frac{1}{2}}+C_2 \\
&\le C(T).
\end{aligned}
$$

Proposition 5 *Assuming (A.1)-(A.3)*

$$\sum_{n=1}^{N}\left(\frac{{\rho_n}^\beta-{\rho_{n-1}}^\beta}{\Delta}\right)^2(t)\Delta\le C(T).$$

it satisfied.

Proof.

Denoting

$$V_n(t)=\left(\frac{b}{\beta}\frac{{\rho_n}^\beta-{\rho_{n-1}}^\beta}{\Delta}+u_n\right)(t)+gt,$$

we can rewrite the equation (2.2) as

$$\frac{d}{dt}V_n(t)=-\frac{(p_n-p_{n-1})(t)}{\Delta}.$$

Multiplying the previous relation by $V_n(t)\Delta$, summing over $n=1,\dots,N$ and integrating with respect to τ from 0 to t, we obtain:

$$\frac{1}{2}\sum_{n=1}^{N}{V_n}^2(t)\Delta=\frac{1}{2}\sum_{n=1}^{N}{V_n}^2(0)\Delta-\int_0^t\sum_{n=1}^{N}V_n(\tau)\frac{p_n-p_{n-1}}{\Delta}(\tau)\Delta d\tau.$$

The first term of the right hand side is bounded by (A.1) and (A.2). Using Propositions 2, 3 and the mean value theorem the second one is estimated as follows,

$$
\begin{aligned}
&-\frac{b}{\beta}\int_0^t\sum_{n=1}^{N}\frac{p_n-p_{n-1}}{\Delta}\frac{{\rho_n}^\beta-{\rho_{n-1}}^\beta}{\Delta}\Delta d\tau \\
&= -\frac{ab\gamma}{\beta^2}\int_0^t\sum_{n=1}^{N}(\rho_{n-1}+\theta_n(\rho_n-\rho_{n-1}))^{\gamma-\beta}\left(\frac{{\rho_n}^\beta-{\rho_{n-1}}^\beta}{\Delta}\right)^2\Delta d\tau,
\end{aligned}
$$

where $0 < \theta_n < 1$,

$$
\begin{aligned}
&- \int_0^t \sum_{n=1}^{N} \frac{p_n - p_{n-1}}{\Delta} u_n \Delta d\tau \\
&= -\frac{a\gamma}{\beta} \int_0^t \sum_{n=1}^{N} (\rho_{n-1} + \theta_n(\rho_n - \rho_{n-1}))^{\gamma-\beta} \frac{{\rho_n}^{\beta} - {\rho_{n-1}}^{\beta}}{\Delta} u_n \Delta d\tau \\
&\leq \frac{ab\gamma}{2\beta^2} \int_0^t \sum_{n=1}^{N} (\rho_{n-1} + \theta_n(\rho_n - \rho_{n-1}))^{\gamma-\beta} \left(\frac{{\rho_n}^{\beta} - {\rho_{n-1}}^{\beta}}{\Delta} \right)^2 \Delta d\tau + C(T),
\end{aligned}
$$

$$
-g \int_0^t \sum_{n=1}^{N} \frac{p_n - p_{n-1}}{\Delta} \tau \Delta d\tau = -g \int_0^t (p_N - p_0)\tau d\tau \leq g \int_0^t p_0 \tau d\tau \leq C(T).
$$

Thus we get

$$
\sum_{n=1}^{N} {V_n}^2(t)\Delta + \int_0^t \sum_{n=1}^{N} \left[(\rho_{n-1} + \theta_n(\rho_n - \rho_{n-1}))^{\gamma-\beta} \left(\frac{{\rho_n}^{\beta} - {\rho_{n-1}}^{\beta}}{\Delta} \right)^2 \right] (\tau)\Delta d\tau \leq C(T).
$$

From this and Proposition 2 we obtain the required estimate.

We are interested in the bound of the density from below.

Proposition 6 *Let (A.1)-(A.3) be satisfied then*

$$
\rho_{n-1}(t) \ \geq \ \underline{\rho}(T),
$$

where $\underline{\rho}(T)$ is a positive constant depending on T.

Proof. Putting

$$
\rho_{K-1} = \max_n \rho_{n-1}, \ (1 \leq K \leq N),
$$

and applying the Proposition 4 and since $\beta - 1 < 0$ we get

$$
{\rho_{K-1}}^{\beta-1} = {\rho_{K-1}}^{\beta-1} \sum_{n=1}^{N} \Delta \leq \sum_{n=1}^{N} {\rho_{n-1}}^{\beta-1}\Delta \leq C(T).
$$

We have also

$$
\begin{aligned}
{\rho_{n-1}}^{\beta-1} &= {\rho_{K-1}}^{\beta-1} + \sum_{k=K}^{n-1} \frac{{\rho_k}^{\beta-1} - {\rho_{k-1}}^{\beta-1}}{\Delta} \Delta \\
&= {\rho_{K-1}}^{\beta-1} + \sum_{k=K}^{n-1} \frac{\beta - 1}{\beta} (\rho_{k-1} + \theta_n(\rho_k - \rho_{k-1}))^{-1} \frac{{\rho_k}^{\beta} - {\rho_{k-1}}^{\beta}}{\Delta} \Delta \\
&\leq C(T) + C \left(\sum_{n=1}^{N} (\rho_{n-1} + \theta_n(\rho_n - \rho_{n-1}))^{-2} \Delta \right)^{\frac{1}{2}} \left(\sum_{n=1}^{N} \left(\frac{{\rho_n}^{\beta} - {\rho_{n-1}}^{\beta}}{\Delta} \right)^2 \Delta \right)^{\frac{1}{2}}.
\end{aligned}
$$

Further, we would like to estimate the second term of the previous inequality. So,

$$\begin{aligned}
&\sum_{n=1}^{N}(\rho_{n-1}+\theta_n(\rho_n-\rho_{n-1}))^{-2}\Delta \\
= &\sum_{n=1}^{N}(\rho_{n-1}+\theta_n(\rho_n-\rho_{n-1}))^{-\beta-1}(\rho_{n-1}+\theta_n(\rho_n-\rho_{n-1}))^{\beta-1}\Delta \\
\leq &\max_n\left[(\rho_{n-1}+\theta_n(\rho_n-\rho_{n-1}))^{-\beta-1}\right]\sum_{n=1}^{N}(\rho_{n-1}+\theta_n(\rho_n-\rho_{n-1}))^{\beta-1}\Delta \\
\leq &\max_n\left[{\rho_{n-1}}^{-\beta-1}\right]\cdot 2\sum_{n=1}^{N}{\rho_{n-1}}^{\beta-1}\Delta \\
\leq &\ C(T)\max_n\left[{\rho_{n-1}}^{-\beta-1}\right].
\end{aligned}$$

From the previous estimates and applying the Proposition 5 it follows that

$${\rho_{n-1}}^{\beta-1}\leq C(T)\left(1+\max_n\left[{\rho_{n-1}}^{-\frac{\beta+1}{2}}\right]\right),$$

From the assumption $\beta<\frac{1}{3}$, we have $\frac{\beta+1}{2}<1-\beta$. And then there is a positive constant $\underline{\rho}(T)$ depending on T such that $\rho_{n-1}(t)\ \geq\ \underline{\rho}(T)$.

The uniqueness of the solution can be prooved by the same manner to [6] . We omit the details.

3 Asymptotic Behavior

We have not yet obtained the result about the asymptotic behavior in the case of the density dependent viscosity. But we have the asymptotic behavior of the density ρ at the free boundary. Let us show that. We consider the difference equation (2.1) at $n=N+1$,

$$\frac{d}{dt}\rho_N+{\rho_N}^2\frac{u_{N+1}-u_N}{\Delta}=0,$$

and the boundary condition (2.5). Then we obtain the following equation,

$$\frac{d}{dt}\rho_N+\frac{\rho_N p_N}{\mu_N}=0.$$

Therefore we get

$$\rho_N(t)=\rho_N(0)\left[1+\frac{b(\gamma-\beta)}{a}\rho_N(0)^{\gamma-\beta}t\right]^{-\frac{1}{\gamma-\beta}}.$$

Next, we consider the case of the constant viscosity. This problem was solved under severe assumptions for $\rho_0(x)$ by M.Okada [1], 1989. Here, we will show the another proof of the asymptotic behavior under less severe assuptions.

We consider the initial boundary problem (1.1)-(1.4) with μ = constant and assumptions $(A.1)'$ and $(A.2)$, where

$$(A.1)' \qquad \rho_0 \in Lip[0,1] \quad \text{and} \quad \rho_0(x) \geq \lambda(x)$$
$$(\lambda(x) \text{ is monotone decreasing and } \int_0^1 \frac{dx}{\lambda(x)} < +\infty),$$

We obtain the existence and the uniqueness of the global weak solution to the above problem by the same method as [1].

By using the same way as getting Proposition 2, we have the following energy estimate, that is, the limit version of Proposition 2.

Proposition 7

$$\begin{aligned} &\int_0^1 \left(\frac{1}{2}u^2 + \frac{a}{\gamma-1}\rho^{\gamma-1} + gy\right)(t)dx + \int_0^t \int_0^1 \mu\rho {u_x}^2 dx d\tau \\ &= \int_0^1 \left(\frac{1}{2}{u_n}^2 + \frac{a}{\gamma-1}\rho^{\gamma-1} + gy\right)(0)dx \ \equiv \ E_0, \\ &\textit{where} \quad y(t,x) = \int_0^x \frac{d\xi}{\rho(t,\xi)} \quad \textit{and} \quad \frac{\partial y}{\partial t} = u. \end{aligned}$$

Using Proposition 7 and the method by I. Straškraba [7], we have the following a priori estimates.

Proposition 8 *There exists a constant C independent of t and such that*

$$\rho(t,x) \leq C \qquad \textit{for} \quad x \in [0,1] \quad \textit{and} \quad t \geq 0.$$

Proof. Rewriting the equation (1.1), we get

$$(3.1) \qquad (\log\rho)_t = \frac{\rho_t}{\rho} = -\rho u_x.$$

Integrating (1.2) with respect to x from x to 1, we have

$$(3.2) \qquad \int_x^1 u_t d\xi - p = -\mu\rho u_x - g(1-x).$$

Then we obtain
$$(\log\rho)_t = \frac{1}{\mu}\int_x^1 u_t d\xi - \frac{p - g(1-x)}{\mu}.$$

Integrating the above with respect to t from t_1 to t_2, we get the following equation,

$$\begin{aligned} \log\rho(t_2) &= \log\rho(t_1) + \int_{t_1}^{t_2} \frac{1}{\mu}\int_x^1 u_t d\xi d\tau - \int_{t_1}^{t_2} \frac{1}{\mu}(p - g(1-x))d\tau \\ &= \log\rho(t_1) + \int_x^1 (u(t_2,\xi) - u(t_1,\xi))d\xi - \int_{t_1}^{t_2} \frac{1}{\mu}(p - g(1-x))d\tau. \end{aligned}$$

If $p - g(1-x) \le 0$ for $t \ge 0$, $\rho \le \left[\frac{g}{a}(1-x)\right]^{\frac{1}{\gamma}}$. If not, there is a $t_2 > 0$ such that $p(t_2) - g(1-x) > 0$. Then there exists a $t_1 \in [0, t_2)$ such that $t_1 > 0$ and $p(t_1) - g(1-x) = 0$ and either $p(t) - g(1-x) \ge 0$ for all $t \in (t_1, t_2)$, or $t_1 = 0$ and $p(t) - g(1-x) \ge 0$ for all $t \in (0, t_2)$.

Therefore we are done with the required estimate.

Proposition 9 *There exists a constant C independent of t and such that*

$$y(t,1) = \int_0^1 \frac{dx}{\rho(t,x)} \le C \qquad \textit{for} \quad t \ge 0.$$

Proof. From (3.1) and (3.2), we have

$$\left(\frac{1}{\rho}\right)_t = u_x = -\frac{1}{\mu\rho}\int_x^1 u_t d\xi + \frac{p - g(1-x)}{\mu\rho}.$$

Integrating the above with respect to t from t_1 to t_2, we obtain the following equation,

$$\frac{1}{\rho(t_2)} = \frac{1}{\rho(t_1)} - \int_{t_1}^{t_2} \frac{1}{\mu\rho(\tau,x)} \int_x^1 u_t d\xi d\tau + \int_{t_1}^{t_2} \frac{p - g(1-x)}{\mu\rho} d\tau.$$

Now, let $x \in [0,1]$ be arbitrary but fixed. If $p(t,x) - g(1-x) \ge 0$ for all $t \ge 0$ then we have that $\rho(t,x) \ge \left[\frac{g}{a}(1-x)\right]^{\frac{1}{\gamma}}$. If that is not held, there is a $t_2 > 0$ such that $p(t,x) - g(1-x) < 0$. Then there exists a $t_1 \in [0, t_2)$ such that $t_1 > 0$ and $p(t_1, x) - g(1-x) = 0$ and either $p(t,x) - g(1-x) \le 0$ for all $t \in (t_1, t_2)$, or $t_1 = 0$ and $p(t,x) - g(1-x) \le 0$ for all $t \in (0, t_2)$. Therefore we have that

$$\frac{1}{\rho(t,x)} \le \left[\frac{g}{a}(1-x)\right]^{-\frac{1}{\gamma}} + \frac{1}{\rho_0(x)} - \int_{t_1}^{t_2} \frac{1}{\mu\rho(\tau,x)} \int_x^1 u_t(\tau,\xi) d\xi d\tau.$$

As $(1-x)^{-\frac{1}{\gamma}}, \rho_0(x) \in L^1(0,1)$, we may estimate

$$J(t) = -\int_0^1 \int_0^t \frac{1}{\mu\rho(\tau,x)} \int_x^1 u_t(\tau,\xi) d\xi d\tau dx.$$

Now, by the equation (1.1), the boundary condition (1.3) and Proposition 7, we get the following estimate.

$$\begin{aligned} J(t) &= -\int_0^1 \int_0^t \left[\frac{1}{\mu\rho(\tau,x)} \int_x^1 u(\tau,\xi) d\xi\right]_\tau d\tau dx + \int_0^1 \int_0^t \left[\frac{1}{\mu\rho(\tau,x)}\right]_\tau \int_x^1 u(\tau,\xi) d\xi d\tau dx \\ &= -\int_0^1 \frac{1}{\mu\rho(t,x)} \int_x^1 u(t,\xi) d\xi dx + \int_0^1 \frac{1}{\mu\rho_0(x)} \int_x^1 u_0(\xi) d\xi dx \\ &\quad + \int_0^1 \frac{1}{\mu} \int_0^t u_x(\tau,x) \int_x^1 u(\tau,\xi) d\xi d\tau dx \end{aligned}$$

$$
\begin{aligned}
&= -\int_0^1 \frac{1}{\mu}\left(\frac{1}{\rho(t,x)}-\frac{1}{\rho_0(x)}\right)\int_x^1 u(t,\xi)d\xi dx - \int_0^1 \frac{1}{\mu\rho_0(x)}\int_x^1 [u(t,\xi)-u_0(\xi)]d\xi dx \\
&\quad + \int_0^1\int_0^t \frac{1}{\mu}\left[u(\tau,x)\int_x^1 u(\tau,\xi)d\xi\right]_x d\tau dx + \int_0^1\int_0^t \frac{1}{\mu}u^2(\tau,x)d\tau dx \\
&\le -\int_0^1 \frac{1}{\mu}\int_0^t \left(\frac{1}{\rho(\tau,x)}\right)_\tau d\tau \int_x^1 u(t,\xi)d\xi dx + C + \int_0^1\int_0^t \frac{1}{\mu}u^2(\tau,x)d\tau dx \\
&= -\int_0^1\int_0^t u_x(\tau,x)d\tau \int_x^1 u(t,\xi)d\xi dx + C + \int_0^1\int_0^t \frac{1}{\mu}u^2(\tau,x)d\tau dx \\
&= -\int_0^t \frac{1}{\mu}\int_0^1 \left[u(\tau,x)\int_x^1 u(t,\xi)d\xi\right]_x dx d\tau - \int_0^t \frac{1}{\mu}\int_0^1 u(\tau,x)u(t,x)dx d\tau \\
&\quad + C + \int_0^1\int_0^t \frac{1}{\mu}u^2(\tau,x)d\tau dx \\
&= -\int_0^t \frac{1}{\mu}\int_0^1 \left[u(\tau,x)u(t,x)-u^2(\tau,x)\right] dx d\tau + C.
\end{aligned}
$$

Here by using $\dfrac{\partial y}{\partial t} = u, \quad \displaystyle\int_0^1 gy dx \le E_0$, we have

$$
\begin{aligned}
&-\int_0^t \frac{1}{\mu}\int_0^1 u(\tau,x)u(t,x)dx d\tau = -\frac{1}{\mu}\int_0^1 [y(t,x)-y_0(x)]\, u(t,x)dx \\
&\le \frac{1}{\mu}\int_0^1 y(t,x)|u(t,x)|dx + C_1 \le \epsilon\int_0^1 y^2(t,x)dx + C_\epsilon \int_0^1 u^2(t,x)dx + C_1 \\
&\le \epsilon y(t,1)\int_0^1 y(t,x)dx + C_2 \le \epsilon \frac{E_0}{g} y(t,1) + C_2.
\end{aligned}
$$

Now, we get

$$
u^2(t,x) = \left(\int_0^x u_\xi(t,\xi)d\xi\right)^2 \le \int_x^1 \frac{d\xi}{\mu\rho}\int_0^1 \mu\rho {u_x}^2 dx \le \frac{1}{\mu}y(t,x)\int_0^1 \mu\rho {u_x}^2 dx.
$$

Then we have

$$
\int_0^1\int_0^t \frac{1}{\mu}u^2 d\tau dx \le \int_0^t \frac{1}{\mu^2}\int_0^1 y(t,x)dx \int_0^1 \mu\rho {u_x}^2 dx d\tau \le \frac{E_0}{\mu^2 g}\int_0^t\int_0^1 \mu\rho {u_x}^2 dx d\tau \le \frac{E_0}{\mu^2 g}.
$$

Therefore we obtain

$$
y(t,1) \le \left(\frac{a}{g}\right)^{\frac{1}{\gamma}} \frac{\gamma}{\gamma-1} + y(0,1) + \epsilon\frac{E_0}{g}y(t,1) + C_2 + \frac{{E_0}^2}{\mu^2 g}.
$$

Here, we choose the value of ϵ such that

$$
\frac{2\epsilon E_0}{g} < 1,
$$

we have the required estimate.

Proposition 10

$$\int_0^1 u^2(t,x)dx \longrightarrow 0 \qquad as\ t \longrightarrow +\infty.$$

Proof. We set

$$\epsilon(t) = \int_{t-1}^t \int_0^1 \mu\rho {u_x}^2 dx d\tau.$$

From Proposition 7, $\epsilon(t) \to 0$ as $t \to +\infty$. Multiplying the equation (1.2) by u, and integrating with respect to x and t from 0 to 1 and s to t $(s < t)$, respectively, we have

$$\int_0^1 \frac{1}{2}u^2(t)dx = \int_0^1 \frac{1}{2}u^2(s)dx + \int_s^t \int_0^1 (pu_x - \mu\rho {u_x}^2 - gu)dx d\tau.$$

Moreover integrating with respect to s from $t-1$ to t, we get

$$\int_0^1 \frac{1}{2}u^2(t)dx = \int_{t-1}^t \int_0^1 \frac{1}{2}u^2(s)dx + \int_{t-1}^t \int_s^t \int_0^1 (pu_x - \mu\rho {u_x}^2 - gu)dx d\tau ds.$$

Here

$$\begin{aligned}
\int_{t-1}^t \int_0^1 \frac{1}{2}u^2(s)dxds &\le \int_{t-1}^t \int_0^1 \frac{1}{2\mu}y(s)dx \int_0^1 \mu\rho {u_x}^2 ds \\
&\le C\epsilon(t) \quad \longrightarrow \quad 0 \qquad \text{as} \quad t \to +\infty.
\end{aligned}$$

$$\begin{aligned}
\left|\int_{t-1}^t \int_s^t \int_0^1 pu_x dx d\tau ds\right| &\le \sup p \int_{t-1}^t \int_s^t \left(\int_0^1 \frac{dx}{\mu\rho}\right)^{\frac{1}{2}} \left(\int_0^1 \mu\rho {u_x}^2 dx\right)^{\frac{1}{2}} d\tau ds \\
&\le C\int_{t-1}^t (\tau - (t-1)) \left(\int_0^1 \mu\rho {u_x}^2 dx\right)^{\frac{1}{2}} d\tau \\
&\le C\left(\int_{t-1}^t (\tau - (t-1))^2 d\tau\right)^{\frac{1}{2}} \left(\int_{t-1}^t \int_0^1 \mu\rho {u_x}^2 dx d\tau\right)^{\frac{1}{2}} \\
&\le C'\sqrt{\epsilon(t)} \quad \longrightarrow \quad 0 \qquad \text{as} \quad t \to +\infty.
\end{aligned}$$

$$\begin{aligned}
\left|\int_{t-1}^t \int_s^t \int_0^1 gu dx d\tau ds\right| &\le g\int_{t-1}^t \int_s^t \int_0^1 \sqrt{\frac{y}{\mu}} \sqrt{\int_0^1 \mu\rho {u_x}^2 dx}\ dx d\tau ds \\
&\le g\int_{t-1}^t \int_s^t \left(\int_0^1 \frac{y}{\mu}dx\right)^{\frac{1}{2}} \left(\int_0^1 \mu\rho {u_x}^2\right)^{\frac{1}{2}} d\tau ds \\
&\le C\sqrt{\epsilon(t)} \quad \longrightarrow \quad 0 \qquad \text{as} \quad t \to +\infty.
\end{aligned}$$

$$\begin{aligned}
\int_{t-1}^t \int_s^t \int_0^1 \mu\rho {u_x}^2 dx d\tau ds &= \int_{t-1}^t (\tau - (t-1)) \int_0^1 \mu\rho {u_x}^2 dx d\tau \\
&\le \epsilon(t) \quad \longrightarrow \quad 0 \qquad \text{as} \quad t \to +\infty.
\end{aligned}$$

Therefore we obtain the required estimate.

We set

$$Q(t) = \int_0^1 (p(t,x) - p_s(x))(\rho(t,x) - \rho_s(x)) \frac{dx}{\rho(t,x)},$$

where $p_s(x)$ and $\rho_s(x)$ are the stationary solutions, that is,

$$p_s(x) = g(1-x), \qquad \rho_s(x) = \left[\frac{g}{a}(1-x)\right]^{\frac{1}{\gamma}}.$$

We have

Proposition 11

$$Q(t) \quad \longrightarrow \quad 0 \qquad as \quad t \to +\infty.$$

Proof. First, we set

$$I(t,x) = \int_0^x (\rho(t,x) - \rho_s(x)) \frac{dx}{\rho(t,x)}.$$

Then, from

$$\frac{\partial I}{\partial x} = \frac{\rho - \rho_s}{\rho}, \quad \frac{\partial I}{\partial t} = \int_0^x \frac{\rho_s}{\rho^2} \rho_t dx = \int_0^x -\rho_s u_x dx = -\rho_s + \int_0^x (\rho_s)_x u dx,$$

we obtain the following.

$$\frac{d}{dt}\int_0^1 uI dx = \int_0^1 (u_t I + u I_t) dx = \int (-p + p_s + \mu\rho u_x)_x I dx - \int_0^1 \rho_s u^2 dx + U(t),$$

where

$$U(t) = \int_0^1 u(t,x) \int_0^x (\rho_s)_x u(t,\xi) d\xi.$$

By using the integration by part and $I(t,0) = 0$, we get

$$\begin{aligned} \frac{d}{dt}\int_0^1 uI dx &= \int_0^1 (-p + p_s + \mu\rho u_x) \frac{\rho - \rho_s}{\rho} dx - \int_0^1 \rho_s u^2 dx + U(t) \\ &= -Q(t) + \int_0^1 \mu u_x (\rho - \rho_s) dx - \int_0^1 \rho_s u^2 dx + U(t). \end{aligned}$$

Integrating with respect to t from $t-1$ to t, we have

$$\begin{aligned} &\int_0^1 uI(t) dx - \int_0^1 uI(t-1) dx \\ &= -\int_{t-1}^t Q(\tau) d\tau + \int_{t-1}^t \int_0^1 \mu u_x (\rho - \rho_s) dx d\tau - \int_{t-1}^t \int_0^1 \rho_s u^2 dx d\tau \int_{t-1}^t U(\tau) d\tau. \end{aligned}$$

Here, as $t \longrightarrow +\infty$, we have

$$\left|\int_0^1 uI(t)dx\right| \leq \sup_{0\leq x\leq 1} I(t,x) \int_0^1 |u|dx \quad \longrightarrow \quad 0,$$

$$\begin{aligned}\left|\int_{t-1}^t \mu u_x(\rho-\rho_s)dxd\tau\right| &\leq C\int_{t-1}^t \sqrt{\int_0^1 \frac{dx}{\rho}}\sqrt{\int_0^1 \mu\rho {u_x}^2 dx}\; d\tau \\ &\leq C'\sqrt{\epsilon(t)} \quad \longrightarrow \quad 0,\end{aligned}$$

$$\left|\int_{t-1}^t\int_0^1 \rho_s u^2 dxd\tau\right| \leq C \sup_{t-1\leq\tau\leq t}\int_0^1 u^2(\tau)dx \quad \longrightarrow \quad 0,$$

$$\begin{aligned}\left|\int_{t-1}^t U(\tau)d\tau\right| &\leq C\int_{t-1}^t\int_0^1 |u|\int_0^x |(\rho_s)_x|\sqrt{\int_0^1 \mu\rho u_x dx}\; d\xi dx d\tau \\ &\leq C'\epsilon(t) \quad \longrightarrow \quad 0.\end{aligned}$$

Hence

$$\int_{t-1}^t Q(\tau)d\tau \quad \longrightarrow \quad 0 \qquad \text{as} \quad t \to +\infty.$$

Next, differentiating $Q(t)$ with respect to t and using (1.1), we get

$$\begin{aligned}\frac{dQ}{dt} &= \int_0^1 p_t(\rho-\rho_s)\frac{dx}{\rho} + \int_0^1 (p-p_s)\frac{\rho_s\rho_t}{\rho^2}dx \\ &= -\int_0^1 [a\gamma\rho^{\gamma-1}(\rho-\rho_s)\rho u_x + (p-p_s)\rho_s u_x]dx.\end{aligned}$$

Noticing that

$$\left|\frac{dQ}{dt}\right| \leq C\int_0^1 |u_x|dx \leq C'\sqrt{\int_0^1 \mu\rho {u_x}^2 dx},$$

and integrating the above equation with respect to t from s to t, we have

$$Q(t) \leq Q(s) + C\int_s^t \sqrt{\int_0^1 \mu\rho {u_x}^2 dx}\; d\tau \leq Q(s) + C\sqrt{t-s}\sqrt{\epsilon(t)}.$$

Moreover integrating the above inequality with respect to s from $t-1$ to t, we get

$$Q(t) \leq \int_{t-1}^t Q(s)ds + C\sqrt{\epsilon(t)} \quad \longrightarrow \quad 0 \qquad \text{as} \quad t \to +\infty.$$

This completes the proof.

Thus we rearch the following.

Proposition 12 *We have*

$$\int_0^1 (p(t,x) - p_s(x))^2 dx \quad \longrightarrow \quad 0 \qquad as \quad t \to +\infty.$$

Proof. From Proposition 8 , we get

$$(p-p_s)(\rho-\rho_s)\frac{1}{\rho} = (p-p_s)^2\frac{\rho-\rho_s}{p-p_s}\frac{1}{\rho} \geq (p-p_s)^2\frac{1}{\sup\frac{dp}{d\rho}}\frac{1}{\sup\rho} \geq \frac{1}{C}(p-p_s)^2.$$

Then

$$\int_0^1 (p-p_s)^2 dx \leq CQ(t) \quad \longrightarrow \quad 0 \qquad \text{as} \quad t \to +\infty.$$

This completes the proof.

Remark 3 *For the sequence* $\{t_n\}_n$ *that* $t_n \to +\infty$, *there exists a subsequence* $\{t_{nk}\}$ *such that* $p(t_{nk}, x) \to p_s(x)$ *a.e.* x *as* $t_{nk} \to +\infty$. *However because* $p_s(x)$ *is unique,* $p(t_n, x) \to p_s(x)$ *as* $t_n \to +\infty$, *really. Therefore* $\rho(t_n, x) \to \rho_s(x)$ *a.e.* x *as* $t_n \to +\infty$. *As* ρ *is bounded, we have also*

$$\int_0^1 |\rho - \rho_s|^q dx \quad \longrightarrow \quad 0 \qquad as \quad t \to +\infty, \quad for \quad 1 \leq \forall q < +\infty.$$

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